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Appearance of a Purely Singular Continuous Spectrum in a Class of Random Schrödinger Operators

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We consider a discrete Schrödinger operator on $l^2(\mathbb{Z})$ with a random potential decaying at infinity as $|n|^{-1/2}$. We prove that its spectrum is purely singular. Together with previous results, this provides simple examples of random Schrödinger operators having a singular continuous component in its spectrum.

KEY WORDS: Singular continuous spectrum; Schrödinger operators.

1. INTRODUCTION

In this paper we study the discrete Schrödinger operator on $l^2(\mathbb{Z})$:

$$(H\psi)_n = \psi_{n+1} + \psi_{n-1} + V(n)\,\psi_n \tag{1}$$

where $V(n)$ is a random potential which decays at infinity as $|n|^{-1/2}$. For the sake of simplicity we assume that

$$\begin{aligned} V_\omega(n) &= \lambda \frac{V_n(\omega)}{|n|^{1/2}}, \qquad n \neq 0 \\ V_\omega(0) &= \lambda V_0(\omega) \end{aligned} \tag{2}$$

where $V_i(\omega)$ are independent identically distributed random variables.

Furthermore we assume that $V_i(\omega)$ is bounded. In Ref. 1 and 2 one can find general results for potentials decaying as $|n|^{-\alpha}$ at infinity. It is

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proven that if $\alpha < 1/2$ then for almost every realization of the potential the spectrum in $[-2, 2]$ is pure point with eigenfunctions decaying at infinity as $\exp(-C|n|^{1-2\alpha})$, whereas for $\alpha > 1/2$, the spectrum in $[-2, 2]$ is purely continuous; by a refinement due to Kotani[3] it is in fact purely absolutely continuous in this case. In the present case ($\alpha = 1/2$) the model has a transition from pure point spectrum to continuous spectrum[1,2]: if $\lambda > \lambda_1$ the spectrum is almost surely pure point with eigenfunctions decaying as a power at infinity and for any interval $K \subset]-2, 2[$ there exists $\lambda_2(K)$ such that for $\lambda < \lambda_2(K)$ the spectrum in K is almost surely purely continuous.

The purpose of this paper is to prove by a refinement of an idea of Pastur[4] that the spectrum is almost surely purely singular; thus when it has a continuous component it is a singular continuous one.

We shall prove the following:

Theorem 1. Let $V_n(\omega)$ be a sequence of bounded identically distributed random variables satisfying $\mathbb{E}(V_n) = 0$. Then with probability 1 the Schrödinger operator on $l^2(\mathbb{Z})$,

$$(H_\omega \psi)_n = \psi_{n+1} + \psi_{n-1} + \lambda \frac{V_n(\omega)}{|n|^{1/2}} \psi_n$$

has no absolutely continuous spectrum.

Using the previous results, this model provides us with a simple example of Schrödinger operators having singular continuous spectrum.

Remark 1. Since the potential goes to zero at infinity the spectrum of H_ω consists of an essential spectrum which is $[-2, 2]$ and possibly a discrete part outside this interval. We are thus left to prove that the spectrum is singular inside $[-2, 2]$ and from now the energy E of the eigenvalue problem will be supposed to lie in this interval.

Remark 2. Our proof does not rely on any ergodic theorem (when Pastur needs in fact the Oseledeč theorem) so that the hypothesis that the random variables are identically distributed can be easily weakened.

Finally we prove a strengthening of Theorem 1 in the case where the random variables have a density on $\mathbb{R}$:

Theorem 2. With the hypothesis of Theorem 1 and assuming furthermore that $\{V_n\}_{n \in \mathbb{Z}}$ have a probability density on $\mathbb{R}$, then the spectral measure of H is almost surely singular with respect to any given (not random) measure.

Proof of the Theorems. Let $\sigma_\omega(dE)$ be the spectral measure of H_ω defined by

$$\sigma_\omega(dE) = \tfrac{1}{2}(\langle \delta_0 | \, E_\omega(dE) \, | \delta_0 \rangle + \langle \delta_1 | \, E_\omega(dE) \, | \delta_1 \rangle) \tag{3}$$

where $E_\omega(dE)$ are the spectral projections associated with H_ω. From the spectral theorem, one knows that for any vector g in $l^2(\mathbb{Z})$, there exists for σ_ω-almost every E, a solution of

$$H_\omega \psi = E\psi \tag{4}$$

such that $g\psi$ is in $l^2(\mathbb{Z})$. Thus if we want to prove that $\sigma_\omega(dE)$ is singular we have to find a vector g in $l^2(\mathbb{Z})$ such that for almost every E (with respect to the Lebesgue measure in $[-2, 2]$) there is no solution of (4) with $g\psi$ in $l^2(\mathbb{Z})$. This actually can be proved only with probability one on the potential, so that we have to prove that

$$\{\exists g \in l^2(\mathbb{Z}), \text{ such that for almost every } \omega, \text{ then for almost every } E, \quad H_\omega \psi = E\psi \Rightarrow g\psi \notin l^2(\mathbb{Z})\} \tag{5}$$

this is by Fubini's lemma, equivalent to

$$\{\exists g \in l^2(\mathbb{Z}), \text{ s.t. for Lebesgue a.e. } E, \exists \Omega_E, \mathbb{P}(\Omega_E) = 1, \quad H_\omega \psi = E\psi \Rightarrow g\psi \notin l^2(\mathbb{Z}) \text{ for any } \omega \text{ in } \Omega_E\} \tag{6}$$

where $\mathbb{P}$ is the probability measure on the potential.

In the case of a stationnary potential (not decaying at infinity) Pastur used the Oseledeč theorem. This theorem together with Furstenberg's theorem ensures that for any energy, then for a.e. ω:

$$H_\omega \psi = E\psi \Rightarrow \lim_{n \to +\infty} \quad \text{or} \quad \lim_{n \to -\infty} \frac{1}{|n|} \log(\psi_n^2 + \psi_{n+1}^2) > 0 \tag{7}$$

It then remains to choose $g_n = 1/n$ for instance to get the result. Furthermore Pastur noticed that since Oseledeč's theorem holds for any energy, we have

$$\{\exists g \in l^2(\mathbb{Z}), \text{ s.t. for all } E, \exists \Omega_E, \mathbb{P}(\Omega_E = 1), \quad H_\omega \psi = E\psi \Rightarrow g\psi \notin l^2(\mathbb{Z}) \text{ for } \omega \in \Omega_E\} \tag{8}$$

which by Fubini's lemma provides us with

$$\{\exists g \in l^2(\mathbb{Z}), \text{ s.t. for } \mathbb{P}\text{-a.e. } \omega, \exists F_\omega, \mu(F_\omega) = 1, \quad H_\omega \psi = E\psi \Rightarrow g\psi \notin l^2(\mathbb{Z}) \text{ for } E \in F_\omega\} \tag{9}$$

for any given (non random) probability measure μ. (9) gives us that $\sigma_\omega(dE)$ is singular with respect to μ. Thus Pastur obtained that almost surely the spectral measure $\sigma_\omega(dE)$ is singular with respect to any given measure μ: $\sigma_\omega(dE)$ is a singular measure (with respect to the Lebesgue measure) which has to "change rapidly" with ω.

In our case we are going to prove (8) in the case where the bounded random variables $\{V_i\}_{i\in\mathbb{Z}}$ have a density on $\mathbb{R}$ which provides us with Theorem 2. More precisely, we need only two of them, say V_0 and V_{-1}, have a density and the others having any common distribution. Now since the absolutely continuous part of the spectral measure is not affected by a local perturbation of the potential[5] we get easily Theorem 1. Thus we are left to prove (8) in the case where the independent bounded random variables V_i have the same distribution for i in $]-\infty, -2[\cup [1, +\infty[$ and a distribution having a density $r(V)\,dV$ for $i=0, -1$.

We first rewrite (4) as

$$\begin{pmatrix}\psi_{n+1}\\ \psi_n\end{pmatrix}=\begin{pmatrix}E-V_n & -1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\psi_n\\ \psi_{n-1}\end{pmatrix} \tag{10}$$

$$=M_n\begin{pmatrix}\psi_n\\ \psi_{n-1}\end{pmatrix}=\tilde{M}_n\begin{pmatrix}\psi_1\\ \psi_0\end{pmatrix} \tag{11}$$

and we shall prove later the following lemma:

Lemma 1. For any E, there exist C, $\alpha(E)$ strictly positive such that

$$\|\tilde{M}_n\|^2 > Cn^{\alpha(E)} \tag{12}$$

with a probability p_n on ω which goes to 1 as n goes to infinity.

Now since M_i are 2×2 matrices with

$$\det M_i = 1$$

we have

$$\|\tilde{M}_n^{-1}\| = \|\tilde{M}_n\|$$

Thus there exist a unit vector u such that

$$\|\tilde{M}_n u\| = \frac{1}{\|\tilde{M}_n\|} \tag{13}$$

and now for any other unit vector u' we have

$$|\det(u, u')| = |\det(\tilde{M}_n u, \tilde{M}_n u')| \leqslant \frac{\|\tilde{M}_n u'\|}{\|\tilde{M}_n\|} \tag{14}$$

which yields

$$\|\tilde{M}_n u'\| \leqslant \|\tilde{M}_n\|^{1/2} \Rightarrow |\det(u, u')| \leqslant \|\tilde{M}_n\|^{-1/2} \tag{15}$$

Together with Lemma 1, (15) ensures that with probability p_n

$$(\psi_{n+1}^2 + \psi_n^2) > C^{1/2} n^{\alpha/2} (\psi_0^2 + \psi_1^2), \qquad n > 1 \tag{16}$$

for any (ψ_0, ψ_1) that does not lie within two angular sectors Δ_1 and Δ_1' with angular measure of order $2C^{-1/4} n^{-\alpha/4}$.

Using once more that $\det M_i = 1$, one can easily check that with probability p_n

$$(\psi_{-n-1}^2 + \psi_{-n}^2) > C'^{1/2} n^{\alpha/2} (\psi_{-1}^2 + \psi_{-2}^2), \qquad n > 2 \tag{17}$$

for any (ψ_{-1}, ψ_{-2}) does not lie within two angular sectors Δ_{-1} and Δ'_{-1} of order $2C'^{-1/4} n^{-\alpha/4}$.

The next step is to check that with a probability near 1 on V_0 and V_{-1}, the relation (10) does not map any point of $\Delta_{-1} \cup \Delta'_{-1}$ in $\Delta_1 \cup \Delta_1'$ so that for any initial condition (ψ_0, ψ_1) at least (16) or (17) will hold.

Let us set

$$\tan \theta_1 = \frac{\psi_1}{\psi_0}, \qquad \tan \theta_{-1} = \frac{\psi_{-1}}{\psi_{-2}}$$

Then we have

$$\tan \theta_1 = V_0 - E - \frac{1}{V - E - (\tan \theta_{-1})^{-1}} \tag{18}$$

Since V_0 and V_{-1} are bounded, there exists a constant K such that

$$0 < \frac{\partial \theta_1}{\partial \theta_{-1}} < K \tag{19}$$

which yields

$$P(\theta_1 \in \Delta_1, \theta_{-1} \in \Delta_{-1}) \leqslant \sup_{\theta_{-1}} P(\theta_1 \in E_1)$$

where E_1 is an interval of angular measure smaller than $|\Delta_1| + K|\Delta_{-1}|$. Now for given θ_{-1} the absolutely continuous measure of V_0 and V_{-1} induces an absolutely continuous measure on θ_1; thus

$$P(\theta_1 \in E_1) = \int_{E_1} \rho_{\theta_{-1}}(\theta_1)\, d\theta_1 \tag{20}$$

Furthermore (19) ensures that $P(\theta_1 \in E_1)$ is continuous as a function of θ_{-1} and thus $P(\theta_1 \in \Delta_1, \theta_{-1} \in \Delta_{-1})$ goes to zero as $|E_1|$ goes to zero, that is, as n goes to infinity. The same proof works also for the others pairs Δ_1, Δ'_{-1},..., thus we finally obtain that with a probability q_n which goes to 1 as n goes to infinity, (16) *or* (17) holds for each (ψ_0, ψ_1).

Furthermore (17) can be easily replaced by

$$\psi^2_{-n+1} + \psi^2_{-n} > C''^{1/2} n^{\alpha/2}(\psi_0^2 + \psi_1^2) \tag{21}$$

for some strictly positive C''.

It then follows by Borel–Cantelli lemma that there exists a strictly increasing sequence n_k such that

$$\forall(\psi_0, \psi_1), \quad \psi^2_{n_k} + \psi^2_{n_k+1} + \psi^2_{-n_k-1} + \psi^2_{-n_k} > C(\omega)\, n_k^{\alpha/2}(\psi_0^2 + \psi_1^2) \tag{22}$$

with $C(\omega)$ almost surely nonzero.

We now choose

$$g_{n_k} = g_{-n_k} = g_{-n_k-1} = g_{n_k+1} = \frac{1}{k^{1/2+\varepsilon}} \tag{23}$$

and $g_n = 0$ otherwise, which gives the result for some ε, assuming Lemma 1. ▌

Proof of the Lemma. The proof relies on the Lagrange method: given an energy E in $]-2, 2[$, the equation

$$\psi_{n+1} + \psi_{n-1} = E\psi_n$$

has two independent solutions $\cos kn$ and $\sin kn$, where k is defined by

$$E = 2\cos k \tag{24}$$

In the equation (10) we make the change of variables

$$\begin{pmatrix} \psi_{n+1} \\ \psi_n \end{pmatrix} = \begin{pmatrix} \cos k(n+1) & \sin k(n+1) \\ \cos kn & \sin kn \end{pmatrix} \begin{pmatrix} A_n \\ B_n \end{pmatrix} \tag{25}$$

and we obtain

$$\begin{pmatrix} A_n \\ B_n \end{pmatrix} = M'_n \begin{pmatrix} A_{n-1} \\ B_{n-1} \end{pmatrix} \tag{26}$$

where

$$\begin{aligned} M'_n &= 1 + \frac{V_n}{(\sin k)\,|n|^{1/2}} \begin{pmatrix} \cos kn \sin kn & \sin^2 kn \\ -\cos^2 kn & -\sin kn \cos kn \end{pmatrix} \\ &= 1 + \frac{V_n}{|n|^{1/2}} T_n \end{aligned} \tag{27}$$

Now since (25) is not singular for $k \neq 0$, π we have only to estimate the norm of

$$\tilde{M}'_n = \prod_{i=1}^{n} M'_i \tag{28}$$

We set

$$X_n = \tilde{M}_n \tilde{M}_n^T \qquad \text{and} \qquad \bar{X}_n = \operatorname{Tr} X_n \tag{29}$$

$\bar{X}_n^{1/2}$ is then a norm for the matrix $\tilde{M}_n$ and we have for fixed p

$$\bar{X}_n = \operatorname{Tr} \left\{ \prod_{i=n-p+1}^{n} \left(1 + \frac{V_i}{|i|^{1/2}} T_i\right) X_{n-p} \left[\prod_{j=n-p+1}^{n} \left(1 + \frac{V_j}{|j|^{1/2}} T_j\right)\right]^T \right\} \tag{30}$$

$$\begin{aligned} &= \bar{X}_{n-p} + \frac{2}{n^{1/2}} \sum_{i=n-p+1}^{n} V_i \operatorname{Tr}(T_i X_{n-p}) \\ &\quad + \sum_{i=n-p+1}^{n} \sum_{j=n-p+1}^{n} \frac{V_i V_j}{n} \operatorname{Tr}(T_i X_{n-p} T_j^T) \\ &\quad + 2 \sum_{\substack{i=n-p+1 \\ i>j}}^{n} \frac{V_i V_j}{n} \operatorname{Tr}(T_i T_j X_{n-p}) + O\left(\frac{1}{n^{3/2}}\right) \cdot \bar{X}_{n-p} \end{aligned} \tag{31}$$

In the last equality, we have to notice that p is fixed and that V_i is bounded which justifies the notation $O(1/n^{3/2})$.

It follows that we have

$$\log \bar{X}_n = \log \bar{X}_{n-p} + Y_n + O\left(\frac{1}{n^{3/2}}\right) \tag{32}$$

where

$$Y_n = \frac{2}{n^{1/2}} \frac{\sum_{i=n-p+1}^{n} V_i \operatorname{Tr}(T_i X_{n-p})}{\bar{X}_{n-p}} + \frac{1}{n} \frac{\sum_{i=n-p+1}^{n} \sum_{j=n-p+1}^{n} V_i V_j \operatorname{Tr}(T_i X_{n-p} T_j^T)}{\bar{X}_{n-p}} + \frac{2}{n} \frac{\sum_{i=n-p+1, i>j}^{n} V_i V_j \operatorname{Tr}(T_i T_j X_{n-p})}{\bar{X}_{n-p}} - \frac{2}{n} \left[\frac{\sum_{i=n-p+1}^{n} V_i \operatorname{Tr}(T_i X_{n-p})}{\bar{X}_{n-p}} \right]^2 \tag{33}$$

Thus there exists a constant C_p such that

$$\log \bar{X}_{np} \geqslant \sum_{i=1}^{n} Y_{ip} - C_p \tag{34}$$

In order to prove that $\bar{X}_{np}$ increases with n, we estimate $\mathbb{E}(Y_{ip} \mid \mathscr{F}_{(i-1)p})$ and $\mathbb{E}(Y_{ip}^2 \mid \mathscr{F}_{(i-1)p})$ where $\mathscr{F}_n$ holds for the σ algebra generated by $\{V_j\}_{1 \leqslant j \leqslant n}$ and $\mathbb{E}(Y \mid \mathscr{F}_n)$ denotes the conditional expectation of Y given $\{V_j\}_{1 \leqslant j \leqslant n}$.

From (33) we have

$$\mathbb{E}(Y_{ip} \mid \mathscr{F}_{(i-1)p}) = \frac{1}{ip} V^2 \frac{\sum_{j=(i-1)p+1}^{ip} \operatorname{Tr} T_j X_{(i-1)p} T_j^T}{\bar{X}_{(i-1)p}} - \frac{2V^2}{ip} \frac{\sum_{j=(i-1)p+1}^{ip} (\operatorname{Tr} T_j X_{(i-1)p})^2}{\bar{X}_{(i-1)p}^2} \tag{35}$$

where V^2 stands for $\mathbb{E}(V_i^2)$ and where we have used that $E(V_i) = 0$ and the independence of V_i and V_j for $i \neq j$.

By the definition of the T_i's (27)

$$\frac{1}{p} \sum_{j=(i-1)p+1}^{ip} \operatorname{Tr}(T_j X_{(i-1)p} T_j^T) = \frac{1}{p \sin^2 k} \sum_{j=(i-1)p+1}^{ip} (u_j, X_{(i-1)p} u_j) \tag{36}$$

and

$$\frac{1}{p} \sum_{j=(i-1)p+1}^{ip} (\operatorname{Tr} T_j X_{(i-1)p})^2 = \frac{1}{p \sin^2 k} \sum_{j=(i-1)p+1}^{ip} (u_j, X_{(i-1)p} u_j^{\perp})^2 \tag{37}$$

where $u_j = (\cos kj, \sin kj)$ and $u_j^{\perp}$ is a unit vector orthogonal to u_j. Then for all $k \neq 0, \pi$, the right-hand side of (36) [respectively (37)] goes (uniformly in $X_{(i-1)p}$) to $(1/2 \sin^2 k)\, \bar{X}_{(i-1)p}$ [respectively, $(1/8 \sin^2 k)(\bar{X}^2_{(i-1)p} - 4)$] as p increases: these limits are easily obtained by taking an average over the angle rather than a summation over j and using the fact that X_n is symmetric with determinant one.

Thus we can choose a finite p, such that

$$\mathbb{E}(Y_{ip} \mid \mathscr{F}_{(i-1)p}) \geqslant \frac{V^2}{6 \sin^2 ki} \qquad \text{for all } i \tag{38}$$

Furthermore

$$\begin{aligned}\mathbb{E}(Y_{ip}^2 \mid \mathscr{F}_{(i-1)p}) &\leqslant \frac{4V^2}{ip} \sum_{j=(i-1)p+1}^{ip} \frac{(\operatorname{Tr} T_j X_{(i-1)p})^2}{\bar{X}^2_{(i-1)p}} + \frac{C'_p}{i^{3/2}} \\ &\leqslant \frac{2V^2}{i \sin^2 k} + \frac{C'_p}{i^{3/2}}\end{aligned} \tag{39}$$

Setting now

$$Z_n = \sum_{i=1}^{n} Y_{ip} - \mathbb{E}(Y_{ip} \mid \mathscr{F}_{(i-1)p}) \tag{40}$$

we have

$$\begin{aligned}\mathbb{E}(Z_n^2) &= \mathbb{E}[\mathbb{E}(Z_n^2 \mid \mathscr{F}_{(n-1)p})] \\ &= \mathbb{E}(Z_{n-1}^2) + \mathbb{E}[\mathbb{E}\{(Y_{np} - \mathbb{E}(Y_{np} \mid \mathscr{F}_{(n-1)p})^2 \mid \mathscr{F}_{(n-1)p}\}] \\ &\leqslant \mathbb{E}(Z_{n-1}^2) + \mathbb{E}[\mathbb{E}(Y_{np}^2 \mid \mathscr{F}_{(n-1)p})] \\ &\leqslant \frac{2V^2}{\sin^2 k} \sum_{i=1}^{n} \frac{1}{i} + C''_p\end{aligned} \tag{41}$$

So using the Tchebichev inequality we get

$$\begin{aligned}P\left(\log \bar{X}_{np} + C_p \leqslant \sum_{i=1}^{n} \frac{V^2}{i\, 12 \sin^2 k}\right) &= P\left(\sum_{i=1}^{n} Y_{ip} \leqslant \sum_{i=1}^{n} \frac{V^2}{i\, 12 \sin^2 k}\right) \\ &\leqslant P\left(Z_n + \sum_{i=1}^{n} \frac{V^2}{i\, 6 \sin^2 k} \leqslant \sum_{i=1}^{n} \frac{V^2}{i\, 12 \sin^2 k}\right) \\ &\leqslant \frac{C''_p + \dfrac{2V^2}{\sin^2 k} \displaystyle\sum_{i=1}^{n} \frac{1}{i}}{\dfrac{1}{144} \dfrac{V^4}{\sin^4 k} \left(\displaystyle\sum_{i=1}^{n} \frac{1}{i}\right)^2}\end{aligned} \tag{42}$$

So that there exist constants C, D such that

$$P(\bar{X}_{np} \leqslant C n^{V^2/(12 \sin^2 k)}) \leqslant D \frac{\sin^2 k}{\log n} \tag{43}$$

which gives Lemma 1 with $\alpha = V^2/(12 \sin^2 k)$.

Furthermore since the right-hand side of (43) decays at infinity like $(\log n)^{-1}$, the subsequence n_k in the proof of the theorems have to increase faster than an exponential; one can choose $n_k = 2^{k^2}$.

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